NORTH-HOLLAND

Monotone Metrics on Matrix Spaces

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ABSTRACT

The study of monotone inner products under stochastic mappings on the space of matrices was initiated by Morozova and Chentsov, motivated by information geometry. They did not show a monotone metric, but proposed several candidates. The main result of the present paper is to provide an abundance of monotone metrics by means of operator monotone functions and to characterize them. It turns out that there is a correspondence between monotone metrics and operator means in the sense of Kubo and Ando. It follows that all proposals of Morozova and Chentsov are indeed monotone metrics.

1. INTRODUCTION

In this paper we are interested in inner products on the space $M_n(\mathbb{C})$ of all complex $n \times n$ matrices. The simplest one among those is certainly the so-called Hilbert-Schmidt inner product, which is given as

$$\langle A, B \rangle = \mathrm{Tr}(A^*B), \tag{1}$$

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where Tr is the usual matrix trace. This inner product is unitarily invariant, that is,

$$\langle A, B\rangle = \langle UAU^*, UBU^*\rangle \tag{2}$$

for every unitary U. The invariance property (2) is so strong that it determines the Hilbert-Schmidt inner product up to a constant multiple. Contrary to (1), we make the inner products depend on a positive definite matrix D of trace 1. We denote by $\mathscr{M}_n$ the set of all positive definite $n \times n$ matrices of trace 1. Assume that for every $A, B \in M_n(\mathbb{C})$, for every $D \in \mathscr{M}_n$ and for every $n \in \mathbb{N}$, a complex number $K_D(A, B)$ is given. If the following conditions hold, then $K_D(\cdot, \cdot)$ will be called metric:

(a) $(A, B) \mapsto K_D(A, B)$ is sesquilinear.
(b) $K_D(A, A) \geqslant 0$, and the equality holds if and only if $A = 0$.
(c) $D \mapsto K_D(A, A)$ is continuous on $\mathscr{M}_n$ for every A.

Monotone metrics will be the center of our interest. Recall that a linear mapping $\mathbf{T}: M_n(\mathbb{C}) \to M_m(\mathbb{C})$ is defined to be stochastic if $\mathbf{T}(\mathscr{M}_n) \subset \mathscr{M}_m$ and $\mathbf{T}$ is completely positive. (In particular, such a $\mathbf{T}$ is trace preserving. Concerning stochastic maps and their importance we refer to [9].) The metric K_D is defined to be monotone if

(d) $K_{\mathbf{T}(D)}(\mathbf{T}(A), \mathbf{T}(A)) \leqslant K_D(A, A)$ for every stochastic mapping $\mathbf{T}: M_n(\mathbb{C}) \to M_m(\mathbb{C})$ and for every $D \in \mathscr{M}_n$ and every A.

The study of metrics in the above sense is motivated by the state space of a quantum system. In quantum mechanics, the state space of an n-level system is identified with the set of all positive n-by-n matrices of trace one; they are called density matrices. The density matrices with strictly positive eigenvalues form a differentiable manifold, on which a differentiable metric determines a Riemannian metric. Monotone Riemannian metrics are important for information-theoretical and statistical considerations on the state space. (Detailed discussion of those topics is out of the scope of the present paper.) The study of monotone metrics was initiated by Morozova and Chentsov [11]; they were motivated by the above-mentioned applications. Morozova and Chentsov have tried to describe the monotone metrics on the space of self-adjoint matrices; their results are summarized in the next section. Actually, they were unable to show any monotone metric, but they proposed several candidates. In the publication [13] we have proved that two of their proposals are indeed monotone.

The goal of the present paper is to provide an abundance of monotone metrics by means of operator monotone functions. Below we mostly impose a symmetry condition on the metrics, but this does not seem to be an essential restriction. It turns out that there is a correspondence between monotone metrics and operator means in the sense of Kubo and Ando [13]. It follows from our results that all proposals of Morozova and Chentsov are indeed monotone.

2. THE WORK OF MOROZOVA AND CHENTSOV

The monograph [5] and the lecture notes [2] establish a differential geometric approach to statistical problems in which the Fisher information metric plays the leading role. This Riemannian metric is the unique Markovian invariant metric on the simplex of probability measures of a finite space (see [5]; cf. [4]). Morozova and Chentsov aimed to extend the geometric approach to the noncommutative (or quantum) setting and proposed the problem of finding monotone Riemannian metrics on the space of density matrices. They achieved the following result in [11].

THEOREM 1 [11]. *Assume that for every $D \in \mathcal{M}_n$ a real bilinear form K'_D is given on the n-by-n self-adjoint matrices such that conditions* (b), (c), *and* (d) *are satisfied for self-adjoint A. Then there exist a positive continuous function $c(\lambda, \mu)$ and a constant C with the following property: If D is diagonal with respect to the matrix units E_{ij}, that is, $D = \sum_i \lambda_i E_{ii}$, then*

$$K'_D(A, A) = C \sum_{i=1}^{n} \lambda_i^{-1} A_{ii}^2 + 2 \sum_{i<j} |A_{ij}|^2 c(\lambda_i, \lambda_j)$$

for every self-adjoint $A = (A_{ij})$. Moreover, c is symmetric in its two variables, $c(\lambda, \lambda) = C\lambda^{-1}$, and $c(t\lambda, t\mu) = t^{-1} c(\lambda, \mu)$.

Since access to [11] might be difficult for many readers, the proof of Theorem 1 is given below in the slightly generalized form of (13). Theorem 1 tells about real monotone metrics on the self-adjoint part of $M_n(\mathbb{C})$. (Indeed, when $\mathcal{M}_n$ is considered as a differential manifold, the tangent vectors may be identified with self-adjoint matrices and the Riemannian metric must be a real bilinear form.) Condition (d) implies the unitary covariance $K'_{U^*DU}(U^*AU, U^*AU) = K'_D(A, A)$, and it is sufficient to describe a monotone metric for a diagonal D. Hence Theorem 1 gives $K'_D(A, A)$ for all D and for all self-adjoint A. Then $K'_D(A, B)$ is determined by polarization.

In our definition of monotone metric, the setting is complexified, which seems to be a simplification for a functional-analytic approach. On the other hand, if K is a monotone metric in the sense of the definition above, then

$$K'_D(A, B) = \tfrac{1}{2}[K_D(A, B) + K_D(B, A)] \qquad (A = A^*, \quad B = B^*)$$

fulfils the assumptions of Theorem 1. The corresponding function c (from Theorem 1) will be called the Morozova-Chentsov function of the metric K_D.

It is remarkable that K'_D is uniquely determined (up to a constant) on the matrices commuting with D. This property is reminiscent of the uniqueness of the monotone Riemannian metric in the probabilistic case [4, 5]. In particular, at the point $D = I/n \in \mathcal{M}_n$ the real monotone metric on $M_n(\mathbb{C})^{sa}$ coincides with the restriction of the Hilbert-Schmidt scalar product.

Morozova and Chentsov were unable to show monotone metrics, but they proposed several candidates corresponding to certain functions $c(\lambda, \mu)$ to use in the setting of Theorem 1:

$$\begin{gathered} \frac{2}{\lambda + \mu}, \quad \left(\frac{2}{\sqrt{\lambda} + \sqrt{\mu}} \right)^2, \quad \frac{\log \lambda - \log \mu}{\lambda - \mu}, \\ \left(\frac{\log \lambda - \log \mu}{\lambda - \mu} \right)^2 \frac{\lambda + \mu}{2}, \quad \frac{\lambda^{2\alpha} + \mu^{2\alpha}}{2(\lambda\mu)^{\alpha + 1/2}} \quad \left(0 \leqslant \alpha \leqslant \tfrac{1}{2}\right). \end{gathered} \tag{3}$$

Below it will be seen that all five functions correspond to a monotone metric, but there are many more possibilities.

3. THE ABUNDANCE OF MONOTONE METRICS

The first examples of monotone metrics were shown in [13]. Set

$$K^r_D(A, B) = \operatorname{Tr}(D^{-1}A^*B). \tag{4}$$

Evidently this is a metric, and to show that condition (d) is satisfied we consider 2-by-2 matrices with matrix entries. For a 2-positive mapping $\mathbf{T}$ the matrix

$$\begin{pmatrix} T(D) & T(A^*) \\ T(A) & T(AD^{-1}A^*) \end{pmatrix}$$

is positive because

$$\begin{pmatrix} D & A^* \\ A & AD^{-1}A^* \end{pmatrix} \geqslant 0.$$

Hence the inequality

$$\mathbf{T}(A)\mathbf{T}(D)^{-1}T(A^*) \leqslant \mathbf{T}(AD^{-1}A^*)$$

follows (cf. [3, 6]). For a trace preserving $\mathbf{T}$ we obtain

$$\mathrm{Tr}\left[\mathbf{T}(D)^{-1}\mathbf{T}(A)^*\mathbf{T}(A)\right] \leqslant \mathrm{Tr}(D^{-1}A^*A), \tag{5}$$

which is exactly the monotonicity property of the inner product K^{r}.

Assume that $D = \mathrm{diag}(\lambda, \mu)$, and let $A = E_{12} + E_{21} \in M_n(\mathbb{C})$. Then

$$K_D^{\mathrm{r}}(A, A) = \frac{1}{\lambda} + \frac{1}{\mu}. \tag{6}$$

So K^{r} is related to the harmonic mean. It appeared in connection with the generalization of the Cramér-Rao inequality in terms of the right logarithmic derivative (see [9, 15]).

Another monotone metric comes from the Bogoliubov inner product, which is also called the Kubo-Mori product, canonical correlation, etc. [13]. Namely,

$$K_D^{\mathrm{Bo}}(A, B) = \int_0^\infty \mathrm{Tr}\, A^*(D+s)^{-1}B(D+s)^{-1}\,ds. \tag{7}$$

Contrary to K^{r}, $K^{\mathrm{Bo}}(A, B)$ is real for self-adjoint A and B.

In this paper we are going to construct monotone metrics by means of operator monotone functions. This approach is very different from the method of Morozova and Chentsov. Its roots are in the quasientropy method developed in [12], and the theory of operator means is an appropriate framework [10]. The theory of operator monotone functions was established in the 1930s by Löwner, and there are several reviews on the subject; for example, [3, 7, 8] are suggested.

Below we denote by bold letters the linear operators acting on $M_n(\mathbb{C})$, which is an inner-product space with the Hilbert-Schmidt scalar product (1). $\mathbf{T}^*$ will stand for the adjoint of $\mathbf{T}$.

Any metric K_D is of the form

$$K_D(A, B) = \langle A, \mathbf{K}_D^{-1}(B)\rangle, \tag{8}$$

where $\mathbf{K}_D$ is a positive (super)operator. The condition (d) of monotonicity is written as

$$\mathbf{T}^*\mathbf{K}_{\mathbf{T}(D)}^{-1}\mathbf{T} \leqslant \mathbf{K}_D^{-1} \tag{9}$$

for any stochastic $\mathbf{T}$.

LEMMA 2. *The metric given by* (8) *is monotone if and only if*

$$\mathbf{T}\mathbf{K}_D\mathbf{T}^* \leqslant \mathbf{K}_{\mathbf{T}(D)}.$$

for every stochastic $\mathbf{T}$.

Proof. (9) is equivalent to $\|\mathbf{K}_{\mathbf{T}(D)}^{-1/2}\mathbf{T}\mathbf{K}_D^{1/2}\| \leqslant 1$, which is again equivalent to $\|\mathbf{K}_D^{1/2}\mathbf{T}^*\mathbf{K}_{\mathbf{T}(D)}^{-1/2}\| \leqslant 1$. The last condition is the statement of the lemma. ∎

Let us introduce some operators on $M_n(\mathbb{C})$:

$$\mathbf{L}_D(A) = DA, \quad \mathbf{R}_D(A) = AD, \quad \mathbf{J}A = A^* \qquad [A \in M_n(\mathbb{C})].$$

In the paper [12] the following inequality was obtained for an operator monotone function $f : \mathbb{R}^+ \to \mathbb{R}$ with $f(0) \geqslant 0$:

$$\mathbf{T}\mathbf{R}_F^{1/2} f(\mathbf{L}_E\mathbf{R}_F^{-1})\mathbf{R}_F^{1/2}\mathbf{T}^* \leqslant \mathbf{R}_{\mathbf{T}(F)}^{1/2} f(\mathbf{L}_{\mathbf{T}(E)}\mathbf{R}_{\mathbf{T}(F)}^{-1})\mathbf{R}_{\mathbf{T}(F)}^{1/2} \tag{10}$$

if E, F are positive definite matrices. In fact, one can apply Theorem 4 of [12] to the Schwarz map $\mathbf{T}^*$ to get

$$\begin{aligned}&\langle f(\mathbf{L}_E\mathbf{R}_F^{-1})\mathbf{T}^*(A)F^{1/2}, \mathbf{T}^*(A)F^{1/2}\rangle\\ &\qquad \leqslant \langle f(\mathbf{L}_{\mathbf{T}(E)}\mathbf{R}_{\mathbf{T}(F)}^{-1})A\mathbf{T}(F)^{1/2}, A\mathbf{T}(F)^{1/2}\rangle\end{aligned}$$

for all A. In the light of Lemma 2, we obtain many monotone metrics from (10).

THEOREM 3. *If $f : \mathbb{R}^+ \to \mathbb{R}^+$ is an operator monotone function, then*

$$\mathbf{K}_D = \mathbf{R}_D^{1/2} f(\mathbf{L}_D \mathbf{R}_D^{-1}) \mathbf{R}_D^{1/2} \tag{11}$$

determines a monotone metric through (8).

The corresponding Morozova-Chentsov function is easy to compute:

$$c(\lambda, \mu) = \frac{\lambda f(\mu/\lambda) + \mu f(\lambda/\mu)}{2\lambda\mu f(\mu/\lambda) f(\lambda/\mu)}. \tag{12}$$

For an operator monotone function f all these give monotone metrics. Let us see some examples. For $f_\beta(t) = t^\beta$ $(0 \leqslant \beta \leqslant 1)$ we obtain

$$c_\beta(\lambda, \mu) = \frac{\lambda_m^{\beta 1-\beta} + \mu^\beta \lambda^{1-\beta}}{2\lambda\mu} = \frac{\lambda^{2\beta-1} + \mu^{2\beta-1}}{2(\lambda\mu)^\beta},$$

which is listed in (3) for $\frac{1}{2} \leqslant \beta \leqslant 1$. If $f_n(t) = [(1 + t^{1/n})/2]^n$ then

$$c_n(\lambda, \mu) = \left(\frac{2}{\lambda^{1/n} + \mu^{1/n}} \right)^n,$$

which is also in (3) for $n = 1, 2$. The condition of Lemma 2 resembles the transformr inequality of operator means. A theory of means of positive operators was developed by Kubo and Ando [10]. Here we do not go into the details concerning operator means, but confine ourselves to the essentials. Operator means are binary operations on positive operators which fulfill the main requirement of monotonicity and the transformer inequality:

(i) If $A \leqslant A'$ and $B \leqslant B'$ then $A\, m\, B \leqslant A'\, m\, B'$.
(ii) $C(A\, m\, B)C^* \leqslant (CAC^*)\, m\, (CBC^*)$.

The key issue of the theory is that operator means are in a one-to-one correspondence with operator monotone functions:

$$A\, m\, B = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2}$$

[A closer look at (10) gives that this inequality is exactly the transformer inequality for a mean of $\mathbf{L}_D$ and $\mathbf{R}_D$.]

THEOREM 4. *Let* $\mathbf{K}^{(1)}$ *and* $\mathbf{K}^{(2)}$ *be families of (super)operators corresponding to monotone metrics according to* (8). *Then for any operator mean* m *the family* $\mathbf{K}_D = \mathbf{K}_D^{(1)} \, m \, \mathbf{K}_D^{(2)}$ *determines another monotone metric.*

Proof. The proof utilizes Lemma 2 and consists in the following chain of inequalities for an arbitrary stochastic $\mathbf{T}$:

$$\mathbf{T}\big(\mathbf{K}_D^{(1)} \, m \, \mathbf{K}_D^{(2)}\big)\mathbf{T}^* \leqslant \big(\mathbf{T}\mathbf{K}_D^{(1)}\mathbf{T}^*\big) \, m \, \big(\mathbf{T}\mathbf{K}_D^{(2)}T^*\big) \leqslant \mathbf{K}_{\mathbf{T}(D)}^{(1)} \, m \, \mathbf{K}_{\mathbf{T}(D)}^{(2)}.$$

The first inequality comes from the transformer inequality, and the second one from monotonicity. ■

Theorem 3 may be deduced from Theorem 4 as follows. $\mathbf{K}_D = \mathbf{R}_D$ determines the metric K^r, which is known to be monotone. $\mathbf{K}_D = \mathbf{L}_D$ induces almost the same metric: more precisely,

$$\langle A, \mathbf{L}_D^{-1}(B)\rangle = \operatorname{Tr} BA^*D^{-1} = K_D^r(B^*, A^*).$$

Hence both $\mathbf{L}_D$ and $\mathbf{R}_D$ induce monotone metrics. Theorem 4 tells us that that $\mathbf{R}_D \, m \, \mathbf{L}_D$ determines a monotone metric whenever m is an operator mean. If m corresponds to the operator monotone function f, then by the commutativity of $\mathbf{L}_D$ and $\mathbf{R}_D$

$$\mathbf{R}_D \, m \, \mathbf{L}_D = \mathbf{R}_D^{1/2} f\big(\mathbf{L}_D \mathbf{R}_D^{-1}\big)\mathbf{R}_D^{1/2}$$

and we obtain the monotone metrics provided by Theorem 3. By the use of operator monotone function we are able to construct monotone metrics, and it is straightforward to ask whether all monotone metrics are of this form.

4. CHARACTERIZATION OF MONOTONE METRICS

If one aims to characterize all monotone metrics, then matrices of all orders should be considered together. Let us emphesis that the monotonicity condition (d) is required also for stochastic mappings between different matrix spaces. The metrics determined by an operator monotone function through (11) are monotone. The converse is the content of the next theorem.

THEOREM 5. *Let* $(A, B) \mapsto K_D(A, B)$ *be a monotone metric. Then there is an operator monotone function* f *such that*

$$K_D(A, B) = \langle A, \mathbf{K}_D^{-1}(B)\rangle \quad \text{and} \quad \mathbf{K}_D = \mathbf{R}_D^{1/2} f\big(\mathbf{L}_D \mathbf{R}_D^{-1}\big)\mathbf{R}_D^{1/2}.$$

Proof. Since the monotonicity condition (d) includes the unitary covariance

$$K_{U^*DU}(U^*AU, U^*BU) = K_D(A, B) \qquad (U \text{ unitary}),$$

we may always assume that $D = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ is diagonal. What we will establish first is the analogue of Theorem 1 in the complex setting:

$$K_D(A, A) = C \sum_{i=1}^{n} \lambda_i^{-1} |A_{ii}|^2 + \sum_{i \neq j} |A_{ij}|^2 d(\lambda_i, \lambda_j), \tag{13}$$

where d is a positive function and C is a positive constant.

We denote by $E_{ij}^{(n)}$ the matrix units in $M_n(\mathbb{C})$. In order to show (13) we shall check that $K_D(E_{ij}^{(n)}, E_{kl}^{(n)}) = 0$ when $(i, j) \neq (k, l)$ and (13) holds true for $A = E_{ij}^{(n)}$. Repeated use of the unitary covariance will be made.

Our first goal is to show that $K_D(E_{12}^{(n)}, E_{12}^{(n)})$ depends only on λ_1 and λ_2. We reach it by constructing a stochastic mapping $\mathbf{T}_1 : M_n(\mathbb{C}) \to \mathscr{M}_3(\mathbb{C})$ and another one $\mathbf{T}_2 : \mathscr{M}_3(\mathbb{C}) \to M_n(\mathbb{C})$ such that

$$\mathbf{T}_1(E_{12}^{(n)}) = E_{12}^{(3)}, \qquad T_1(D) = \operatorname{diag}(\lambda_1, \lambda_2, 1 - \lambda_1 - \lambda_2) \equiv D'$$

and $\mathbf{T}_2$ acts as the inversion on these matrices. In this way, we obtain by a double use of the monotonicity that the scalar product $K_D(E_{12}^{(n)}, E_{12}^{(n)})$ is a function d of λ_1 and λ_2 independently of n. $\mathbf{T}_1$ and $\mathbf{T}_2$ can be given explicitly by the following formulas:

$$\mathbf{T}_1(E_{ij}^{(n)}) = \begin{cases} E_{ij}^{(3)} & \text{if} \quad i, j = 1, 2, \\ E_{33}^{(3)} & \text{if} \quad i = j = 3, 4, \dots, n, \\ 0 & \text{otherwise}. \end{cases}$$

$$\mathbf{T}_2(E_{ij}^{(3)}) = \begin{cases} E_{ij}^{(n)} & \text{if} \quad i, j = 1, 2, \\ \dfrac{1}{1 - \lambda_1 - \lambda_2} \displaystyle\sum_{k=3}^{n} \lambda_k E_{kk}^{(n)} & \text{if} \quad i = j = 3, \\ 0 & \text{otherwise}. \end{cases}$$

So $K_D(E_{12}^{(n)}, E_{12}^{(n)}) = d(\lambda_1, \lambda_2)$, and by unitary covariance we have

$$K_D\big(E_{ij}^{(n)}, E_{ij}^{(n)}\big) = d(\lambda_i, \lambda_j)$$

for every $i \neq j$.

Concerning the diagonal A's, we may refer to the uniqueness of the monotone metric in the probabilistic case; we conclude that

$$K_D\left(E_{ii}^{(n)}, E_{ii}^{(n)}\right) = C\lambda_i^{-1},$$

and $d(\lambda, \lambda) = C\lambda^{-1}$ comes from the unitary covariance again. (In fact, if D and A commute, then $K_D(A, A) = C\,\mathrm{Tr}(D^{-1}A^2)$ is a matricial reformulation of the well-understood uniqueness result from probability theory; cf. [4].)

To show that $K_D(E_{ij}^{(n)}, E_{kl}^{(n)}) = 0$ when $(i, j) \neq (k, l)$ we apply the polarization identity. So it suffices to prove that

$$K_D\left(E_{ij}^{(n)} + sE_{kl}^{(n)}, E_{ij}^{(n)} + sE_{kl}^{(n)}\right) = K_D\left(E_{ij}^{(n)} - sE_{kl}^{(n)}, E_{ij}^{(n)} - sE_{kl}^{(n)}\right)$$

for $s = 1, i$. This follows from the unitary covariance, because it is easy to find a diagonal unitary U such that $U^*E_{ij}^{(n)}U = -E_{kl}^{(n)}$.

By means of the stochastic mappings $D \mapsto D \otimes I/n$ one can see that $d(\lambda/n, \mu/n) = nd(\lambda, \mu)$ and more generally $d(q\lambda, q\mu) = d(\lambda, \mu)/q$ for a positive rational number q whenever $0 < q\lambda, q\mu, \lambda, \mu < 1$. This is the point where the continuity assumption on the metric is needed and we establish that d is homogeneous of order -1. (Morozova and Chentsov proved the homogeneity of the function c of Theorem 1 exactly the same way in Lemma 7.3 of [11].)

If we choose $f(t) = 1/d(t, 1)$, then $d(\lambda_i, \lambda_j) = 1/\lambda_j f(\lambda_i/\lambda_j)$, and by means of this function the monotone metric can be written in terms f exactly in the form stated in the theorem. What we have to prove is that f is operator monotone.

We observe that the definition of $K_D(A, B)$ may be extended to arbitrary positive definite X as

$$K_X(A, B) = (\mathrm{Tr}\, X) K_D(A, B), \qquad D = \frac{X}{\mathrm{Tr}\, X},$$

while the monotonicity property (d) remains valid for arbitrary positive definite X.

We now choose a particular stochastic mapping $\mathbf{T}$. Set

$$\mathbf{T} : X \equiv \begin{pmatrix} X_1 & A \\ B & X_2 \end{pmatrix} \mapsto \begin{pmatrix} (X_1 + X_2)/2 & 0 \\ 0 & (X_1 + X_2)/2 \end{pmatrix},$$

which is a kind of partial trace on block matrices. With this choice the condition $\mathbf{T}\mathbf{K}_X\mathbf{T}^* \leqslant \mathbf{K}_{\mathbf{T}(X)}$ yields that

$$Y \mapsto f(\mathbf{L}_Y \mathbf{R}_Y^{-1})\mathbf{R}_Y \tag{14}$$

is a concave mapping, or equivalently,

$$Y \mapsto \left(Y \otimes (Y^{-1})^t\right)(I \otimes Y^t) \tag{15}$$

is concave for positive definite Y. We write down (15) for a block matrix

$$Y = \begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix};$$

then we observe that concavity of (16) implies the concavity of the mapping

$$(Y_1, Y_2) \mapsto f\left(Y_1 \otimes (Y_2^{-1})^t\right)(I \otimes Y_2^t). \tag{16}$$

Now the choice $Y_2 = I$ gives that the mapping $Y_1 \mapsto f(Y_1)$ must be concave. The operator concavity of f is equivalent to the operator monotonicity of f (cf. [8]). ■

To get rid of the constant C in (13), one can normalize the monotone metrics such a way that $K_D(A, A) = \mathrm{Tr}(D^{-1}AA^*)$ for commuting A and D. This normalization corresponds to $f(1) = 1$. Under this restriction there exists a one-to-one (affine) correspondence between operator means and operator monotone functions $f: \mathbb{R}^+ \to \mathbb{R}^+$. Therefore we arrive at the following.

COROLLARY 6. *There exists a one-to-one correspondence between the monotone metrics satisfying* $K_D(I, I) = \mathrm{Tr}(D^{-1})$ *and the operator means. The correspondence is determined by* (11).

Note that this correspondence is nonaffine.

5. THE SYMMETRY CONDITION

Several metrics may have the same Morozova-Chentsov function. For example,

$$\langle A, \mathbf{R}_D^{-1}(B)\rangle = \mathrm{Tr}(D^{-1}A^*B) \quad \text{and} \quad \langle A, \mathbf{L}_D^{-1}(B)\rangle = \mathrm{Tr}(BA^*D^{-1})$$

give the same

$$\langle A, \mathbf{R}_D^{-1}(A)\rangle = \mathrm{Tr}\, D^{-1}A^2 = \langle A, \mathbf{L}_D^{-1}(A)\rangle$$

for a self-adjoint A. To avoid this kind of ambiguity, we pose a symmetry condition:

(e) $K_D(A, B) = K_D(B^*, A^*)$.

Every metric can clearly be symmetrized, and the symmetrization of K^r yields $K_D^s(A, B) = \operatorname{Tr} D^{-1}(A^*B + BA^*)/2$.

THEOREM 7. *Equation* (11) *determines a symmetric metric if and only if* $f(t) = tf(t^{-1})$ *holds for the function* f. *Any symmetric monotone metric may be obtained from* (11) *by the choice of a suitable function* f.

Proof. The symmetry condition (e) reads as

$$\mathbf{J}\mathbf{K}_D^{-1}\mathbf{J} = \mathbf{K}_D^{-1}, \tag{17}$$

where $\mathbf{J}$ is the conjugate linear operator $A \mapsto A^*$. When K_D is of the form (11), then (17) is equivalent to

$$f(\mathbf{L}_D\mathbf{R}_D^{-1})\mathbf{R}_D = f(\mathbf{R}_D\mathbf{L}_D^{-1})\mathbf{L}_D$$

or

$$f(\mathbf{L}_D\mathbf{R}_D^{-1}) = f(\mathbf{R}_D\mathbf{L}_D^{-1})\mathbf{L}_D\mathbf{R}_D^{-1}$$

(since $\mathbf{J}\mathbf{R}_D\mathbf{J} = \mathbf{L}_D$). The last condition (on f) is equivalent to $f(x) = xf(x^{-1})$. ∎

To see an example, consider

$$f_\alpha(x) = \frac{2x^{\alpha+1/2}}{1 + x^{2\alpha}} \qquad \left(0 \leqslant a \leqslant \tfrac{1}{2}\right). \tag{18}$$

Then

$$f_\alpha(x)^{-1} = \tfrac{1}{2}x^{-\alpha-1/2} + \tfrac{1}{2}x^{\alpha-1/2}$$

is clearly operator monotone decreasing, so f_α is operator monotone. The corresponding metric was conjectured to be monotone by Moroza and Chentsov. The operator monotone function

$$f(x) = \frac{x-1}{\log x} = \int_0^1 x^t\, dt \tag{19}$$

yields the Bogoliubov metric, which corresponds to the logarithmic mean. Other monotone metrics may be obtained from the recursively defined double sequence

$$g_{n+1}(x) = \frac{g_n(x) + h_n(x)}{2}, \qquad h_{n+1}(x) = \sqrt{g_{n+1}(x) h_n(x)} \quad (20)$$

(in the way of construction of means in Theorem 6.2 of [10]) from arbitrary initial operator monotone data g_1 and h_1. The initial conditions

$$g_1(x) = \sqrt{x}, \qquad h_1(x) = \frac{2x}{x+1}$$

yield the (joint) limit

$$\frac{x-1}{\log x} \frac{2\sqrt{x}}{1+x}.$$

Applying (20) once more with

$$g_1(x) = \frac{x-1}{\log x} \quad \text{and} \quad h_1(x) = \frac{x-1}{\log x} \frac{2\sqrt{x}}{1+x},$$

we arrive at the operator monotone function

$$\left(\frac{x-1}{\log x} \right)^2 \frac{2}{1+x}. \quad (21)$$

The operator mean of the function (21) looks unusual, but the corresponding Morozova-Chentsov function is the fourth item in the list (3), and so it is related to a monotone metric. Hence all metrics conjectured in [11] have been proven to be monotone.

Corollary 8. *There is a one-to-one correspondence between symmetric operator means and symmetric monotone metrics satisfying the condition* $K_D(I, I) = \mathrm{Tr}\, D^{-1}$.

COROLLARY 9. *Among the symmetric monotone metrics satisfying the condition* $K_D(I, I) = \operatorname{Tr} D^{-1}$ *there are smallest and largest. They have the Morozova-Chentsov functions*

$$\frac{2}{\lambda + \mu}, \qquad \frac{\lambda + \mu}{2(\lambda\mu)},$$

respectively.

Proof. According to the Kubo-Ando theory the harmonic operator mean is the smallest symmetric mean and the arithmetic is the largest; see [10]. The inverse in (8) changes the order relation between means and the corresponding metrics. ∎

The smallest metric corresponds to the arithmetic mean and given as

$$K_D^{\min}(A, B) = \operatorname{Tr} A^* G, \qquad \text{where} \quad DG + GD = 2B. \tag{22}$$

The largest is related to the harmonic mean and is of the form

$$K_D^{\max}(A, B) = \operatorname{Tr} D^{-1}(A^*B + BA^*)/2. \tag{23}$$

6. DISCUSSION

In quantum Cramér-Rao inequalities, where a lower bound is sought for $\operatorname{Tr}(DA^2)$ in terms of the inverse of the norm of a so-called logarithmic derivative, the smallest metrics $K^{\min}$ is the most informative [9]. This metric appeared in [15] in the form of (22). We note that monotonicity of $K^{\min}$ is equivalent to the property

$$\operatorname{Tr}[\mathbf{T}(A^*)H] \leqslant \operatorname{Tr}(A^*G^2),$$

where $\mathbf{T}$ is a stochastic mapping and G and H are given by the equations

$$DG + GD = 2A, \qquad \mathbf{T}(D)H + H\mathbf{T}(D) = 2\mathbf{T}(A).$$

$K^{\min}$ is the metric which has been intensively studied by Uhlmann in several papers from another viewpoint; see for example [14]. This metric is the infinitesimal form of the so-called Bures distance and may be continued to the boundary of the manifold $\mathscr{M}_n$. There it reduces to the Fubini-Study Riemannian metric. The maximal metric does not admit continuous extension to noninvertible positive semidefinite matrices. A detailed analysis of related questions is left to a later publication.

Finally, we make some comments about positivity of $\mathbf{T}$ in connection with the monotonicity condition

$$K_{\mathbf{T}(D)}(\mathbf{T}(A), \mathbf{T}(A)) \leqslant K_D(A, A). \tag{24}$$

In fact, (24) holds true if $\mathbf{T}(\mathscr{M}_n) \subset \mathscr{M}_m$, the metric K is monotone and $\mathbf{T}$ is 2-positive. (Then $\mathbf{T}^*$ is Schwarz positive and the proof of Theorem 4 works.) In other words, if a metric is monotone under completely positive mappings, it is so under 2-positive ones.

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